MEASUREMENT OF THE STATISTICAL PARAMETERS OF A NONSTATIONARY STOCHASTIC PROCESS

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SUMMARY

The object of the research described in this thesis is to devise and implement a system to measure the first two moments of a nonstationary stochastic process. More precisely, the moments to be measured are the mean, $M(t)$, and the covariance, $R(t, t + \tau)$, of the process $X(t)$. Essentially the system involves reducing several continuous sample functions of the process to discrete sampled functions using analog-to-digital conversion equipment and then performing the necessary arithmetic calculations on a digital computer. The effectiveness of the measuring system is checked by measuring several stochastic processes with known mean and covariance.

CHAPTER I

INTRODUCTION

This study is part of a larger problem, namely, that of producing in the laboratory a nonstationary process which is an analog of a particular process in nature. This general problem may be attacked with the following procedure: record the process to be simulated; measure the first two moments of this process; synthesize a linear time-varying filter which will operate on stationary, Gaussian noise to give a stochastic process with the desired moments; and measure the resulting process to check its validity. Two major problems are involved here: 1) devise a system to measure the moments of the natural process and the simulated process; and 2) devise a procedure for synthesizing the appropriate time-varying filter. This study is directly concerned with only the first problem since Webb (1) describes an adequate filter synthesis procedure. In fact, the nonstationary processes used to test the measuring system (as described in Chapter VI) are generated using time-varying filters realized by the techniques of Webb.

Nonstationary processes produced in the laboratory can be extremely useful in analog computer Monte Carlo studies. Consider for example the problem of obtaining statistical data on the performance of a rocket moving at a constant rate in a vertical direction in the presence of buffeting winds which can be regarded as statistical in nature. The required data may be obtained by simulating the rocket system on an analog computer and making several computational runs. Assuming that the wind at a given altitude is a stationary process, the fact that the rocket is moving vertically through the atmosphere gives rise to a nonstationary process. This process might be simulated for use in the analog computer model of the rocket system using the procedure previously described.

Considerable work has been done with Monte Carlo techniques on systems involving stationary processes. Using the measuring system described in this study and the synthesis procedure of Webb, many of these applications may be extended to include nonstationary processes.

In the following pages the exact quantities to be measured are defined, the measuring system is described in detail, the statistical error associated with the measurements is formulated, and the results of applying several test processes to the measuring system are analyzed.

CHAPTER II

QUANTITIES TO BE MEASURED

The purpose of the measuring system is to determine the first two moments of a nonstationary stochastic process. The moments to be measured are the mean, $M(t)$, and the covariance, $R(t, t + \tau)$, of the nonstationary process $X(t)$. These quantities are defined by

$$M(t_i) = E(X_i) \tag{2.1}$$

and

$$R(t_i, t_j) = E\left\{[X_i - E(X_i)][X_j - E(X_j)]\right\}$$

$$= E(X_i X_j) - E(X_i)E(X_j) \tag{2.2}$$

where X_i and X_j are the vector values of $X(t)$ at $t = t_i$ and $t = t_j$ respectively, and $E(\cdot)$ indicates the expected value.

The moments calculated by the measuring system are the discrete equivalents $M(t_i)$ and $R(t_i, t_j)$ of the continuous variables $M(t)$ and $R(t, t + \tau)$ where

$$t = t_i, \; i = 0,1,\ldots,m$$

and

$$\tau = t_j - t_i,\ j \geq i.$$

To evaluate precisely these quantities defined by equations (2.1) and (2.2) for some particular values of the independent variables t_i and t_j requires either an integration across the ensemble (the integrand of which is not available when the process is given as an ensemble of sample functions) or a summation of an uncountable number of samples which is impossible. Therefore, the calculations performed by the measuring system must be based upon estimates of the actual variables. A convenient estimate of $M(t_i)$ is

$$M_E(t_i) = \frac{1}{N} \sum_{n=1}^{N} \overset{n}{X}_i \tag{2.3}$$

where superscript n and subscript i indicate that $\overset{n}{X}_i$ is the value of the n^{th} sample function at $t = t_i$, and N is the number of sample functions over which the average is taken. (Figure 1 is a pictorial representation of such an ensemble of sample functions.) Using the same notation, a convenient estimate of $R(t_i, t_j)$ is

$$R_E(t_i, t_j) = \frac{1}{N} \sum_{n=1}^{N} (\overset{n}{X}_i \overset{n}{X}_j) - M_E(t_i) M_E(t_j). \tag{2.4}$$

Clearly the estimated quantities defined by equations (2.3) and (2.4) approach the true values if N is allowed to increase without limit. The statistical error introduced by truncating the summations in equations (2.3) and (2.4) after a finite number of terms is formulated in the next chapter.

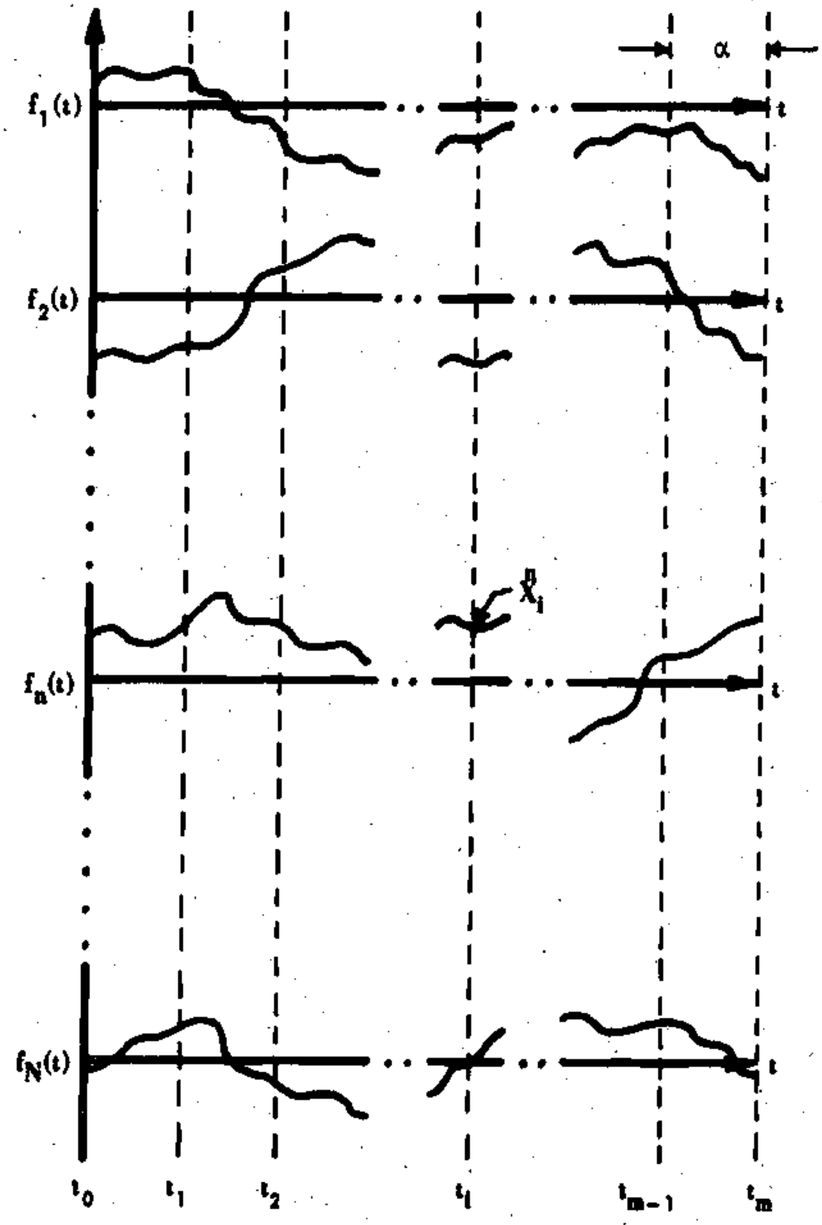

Figure 1. Ensemble of Sample Functions Representing the Non-Stationary Stochastic Process x(t).

CHAPTER III

MEASURING SYSTEM

This chapter describes the actual measuring system resulting from this study. In particular, the first section considers the requirements of the system; the second section deals with the functions of the various components in meeting these requirements; the third section considers some pertinent details in the system design; and the fourth section gives an analysis of the error associated with the measurements performed by the system.

System Requirements

Recall that the overall purpose of the measuring system is to calculate the mean and covariance of a nonstationary process for various specified values of t and τ using the estimates defined by equations (2.3) and (2.4). To perform this function the system must first sample a given number of sample functions from the process, say N sample functions, a given number of times, say $m + 1$ times, with a given spacing between the samples, say α seconds. Also it is necessary that the first sample of each sample function be taken at some fixed value of the time base so that corresponding samples of each sample function will be referenced identically to the time base. Once the sampling is completed, the necessary arithmetic calculations described by (2.3) and (2.4) must be performed. The results of these manipulations then yield the desired information, namely, calculated values for $M(t)$ and $R(t, t + \tau)$ for specified values of t and τ.

Detailed requirements on bandwidth, voltage levels, etc. are largely predetermined by designing the system around certain pieces of existing equipment--i.e., an analog-to-digital converter assembly consisting of a digital voltmeter, a serializer, and a paper tape punch. Of these three pieces of equipment, the serializer is the slowest. The maximum rate for consistently accurate operation of the serializer is one sample per second; consequently, it is necessary to restrict processes to be measured to a bandwidth of approximately two radians per second.

In order to sample precisely (with respect to the time base) and to have some degree of flexibility with respect to sampling rates, it is necessary that the system include a device for generating sampling commands external to the analog-to-digital equipment. In particular this device must generate voltage pulses at a maximum rate of four per second (since the serializer requires four pulses per sample) with a minimum width of one millisecond and any reasonable voltage level above one volt (the digital voltmeter has a very low threshold and the serializer has a variable threshold).

Also it is necessary that some means be provided for generating nonstationary processes with known mean and covariance for checking the measuring system. Maximum voltage levels for such processes are not critical since the digital voltmeter has four voltage ranges available, namely, $\pm$ 1, 10, 100, and 1000 volts. Such processes must of course meet the maximum bandwidth requirement of two radians per second and must be generated such that they can be synchronized with the sample pulses.

System Operation

The complete experimental system is shown in block diagram form in

Figure 2. All example processes used in this presentation are generated on the analog computer along with the sample pulses, s(t), which are used to trigger the analog-to-digital converter. This conveniently allows synchronization of s(t) and X(t) with respect to a common time base.

Actual sampling of the ensemble is performed by a digital voltmeter at the command of the sample pulses. A binary coded output provided by the digital voltmeter is further modified by a diode matrix serializer and then punched on paper tape. This modification by the serializer involves changing the coded digital voltmeter output from a parallel representation (the three coded digits and sign bit appear simultaneously) to a series representation (the three digits and sign bit appear as a sequence of pulses). The series representation is required by the punching equipment.

Arithmetic calculations on the sampled ensemble, $X^*(t)$, as indicated by equations (2.3) and (2.4) are then performed on the Burroughs B-5000 digital computer. Using Figure 1 it is possible to "visualize" these calculations. In this figure note that $f_1, \ldots, f_N$ are sample functions from some process. The vertical dotted lines $t_0, \ldots, t_m$ represent the times at which such an ensemble might be sampled. In this case the sample functions appear individually rather than simultaneously, since the system can take only one sample at a time. To calculate $M(t_i)$ for some fixed i, the digital computer simply sums the samples taken on all N sample functions at t_i and then divides by N. To calculate $R(t_i, t_j)$, for fixed i and j, the digital computer first multiplies the samples taken from each sample function at t_i and t_j and then sums the products and divides by N.

The counter-controller is used to automatically reset the computer, i.e., return the time reference to zero, when a preset number of samples

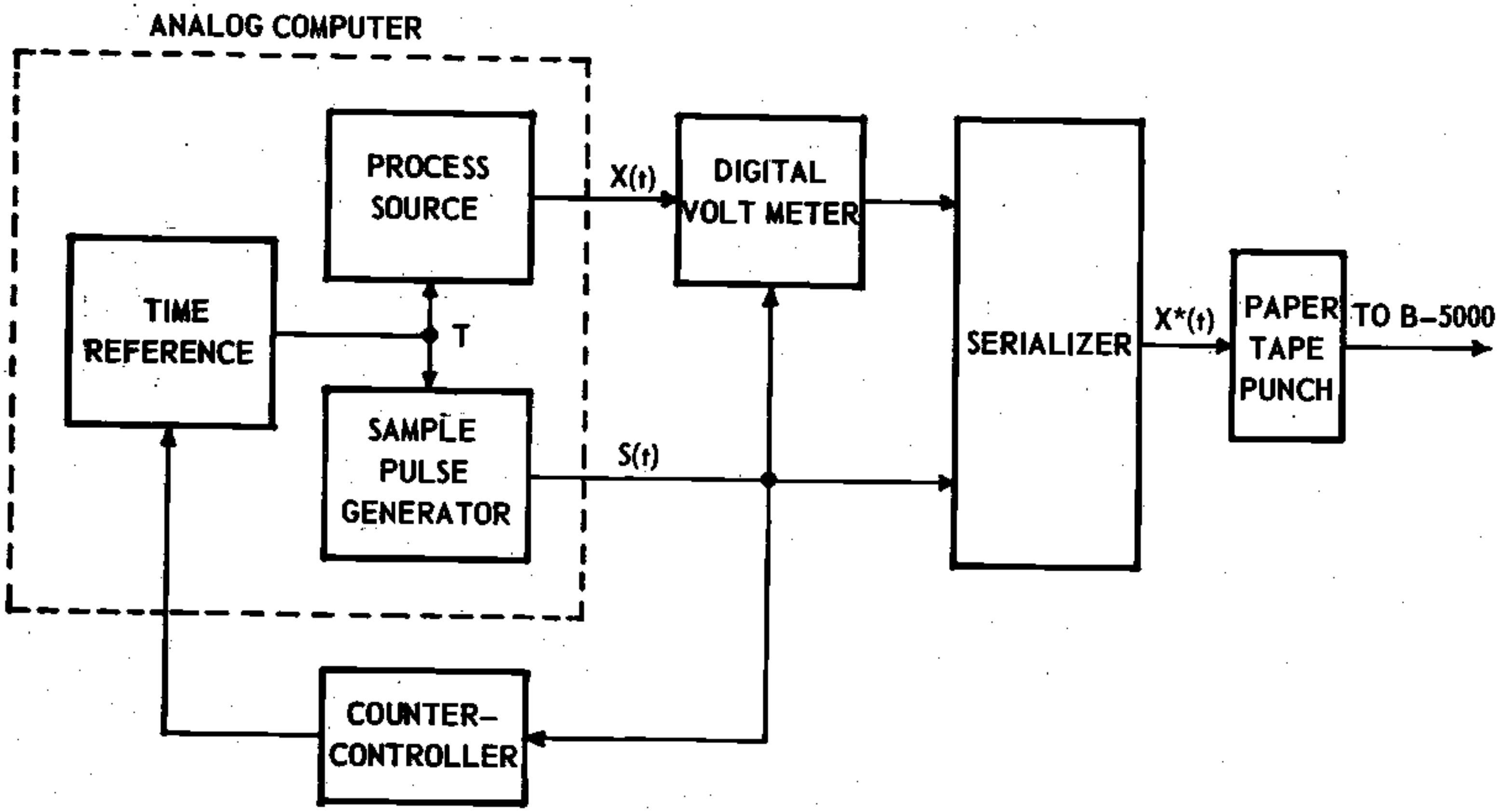

Figure 2. Measuring System.

is reached. The next sample function from the ensemble is then initiated manually by placing the computer in the compute mode. This allows automatic control of the length of the sample functions from a given process.

Before measuring a process, three parameters of the measuring system should be assigned values: 1) the sampling interval, α; 2) the number of samples per sample function, $m + 1$; and 3) the number of sample functions over which the averaging described by Equations (2.3) and (2.4) will be taken, N. (See Figure 1 for a pictorial description of these parameters.) Values for m and α are chosen on the basis of the expected shape of the functions $M(t)$ and $R(t, t + \tau)$. α is chosen such that the distance between the discrete points at which $M(t)$ and $R(t, t + \tau)$ are measured will be close enough together to approximate the ideal smooth curves. Once α is fixed, the length of time over which $M(t)$ and $R(t, t + \tau)$ are measured is determined by m. N is chosen considering that the accuracy of the measured points $M(t_i)$ and $R(t_i, t_j)$ increases with N and also that N must be kept within a reasonable size to expedite the measuring procedure . A compromise between the two requirements placed on N may be reached with the aid of the formulation relating N to statistical error developed later in this chapter.

System Design

All of the components discussed in the previous section and shown in Figure 2 are, for the most part, standard electronic and electro-mechanical gear. There are, however, two exceptions: 1) the sample-pulse generator, and 2) the time-varying filters used to generate the nonstationary test processes. Both of these devices may be constructed using a general-purpose analog computer. The standard components are described in

Appendix II, and the time-varying filters are described in detail in Appendix I. This section then is devoted mainly to the design details of the sample pulse generator.

Figure 3 shows the analog computer program used to generate the sample pulses, and Figure 4 shows some typical waveforms in this generator. In general terms, the generator consists of a free-running multivibrator (with a variable duty cycle) which drives a one-shot multivibrator (with variable output pulse width). Note that the two important variable characteristics of the final output pulses are the duty cycle and the sample pulse width. Also note that these characteristics may be varied independently. The duty cycle, τ, (referred to previously as the measuring system parameter α) is determined by the computer program parameter K_1:

$$\tau = K_1 \ (1011).$$

The sample pulse width, τ_1, is determined by the program parameter K_2:

$$\tau_1 = .0025 \ \ln \left(\frac{100}{K_2}\right) \ .$$

Because of the sensitivity of the voltmeter which is triggered by the sample pulses, it is necessary to remove the small negative bias which appears at the output of amplifier 4 during the "off" portion of its cycle. (This bias is caused by the voltage drop across the forward biased diode in the feedback.) Amplifier 5 is used to remove the bias.

The "preset switch" indicated in the program must be closed momentarily before each sample function is run to ensure that the first pulse

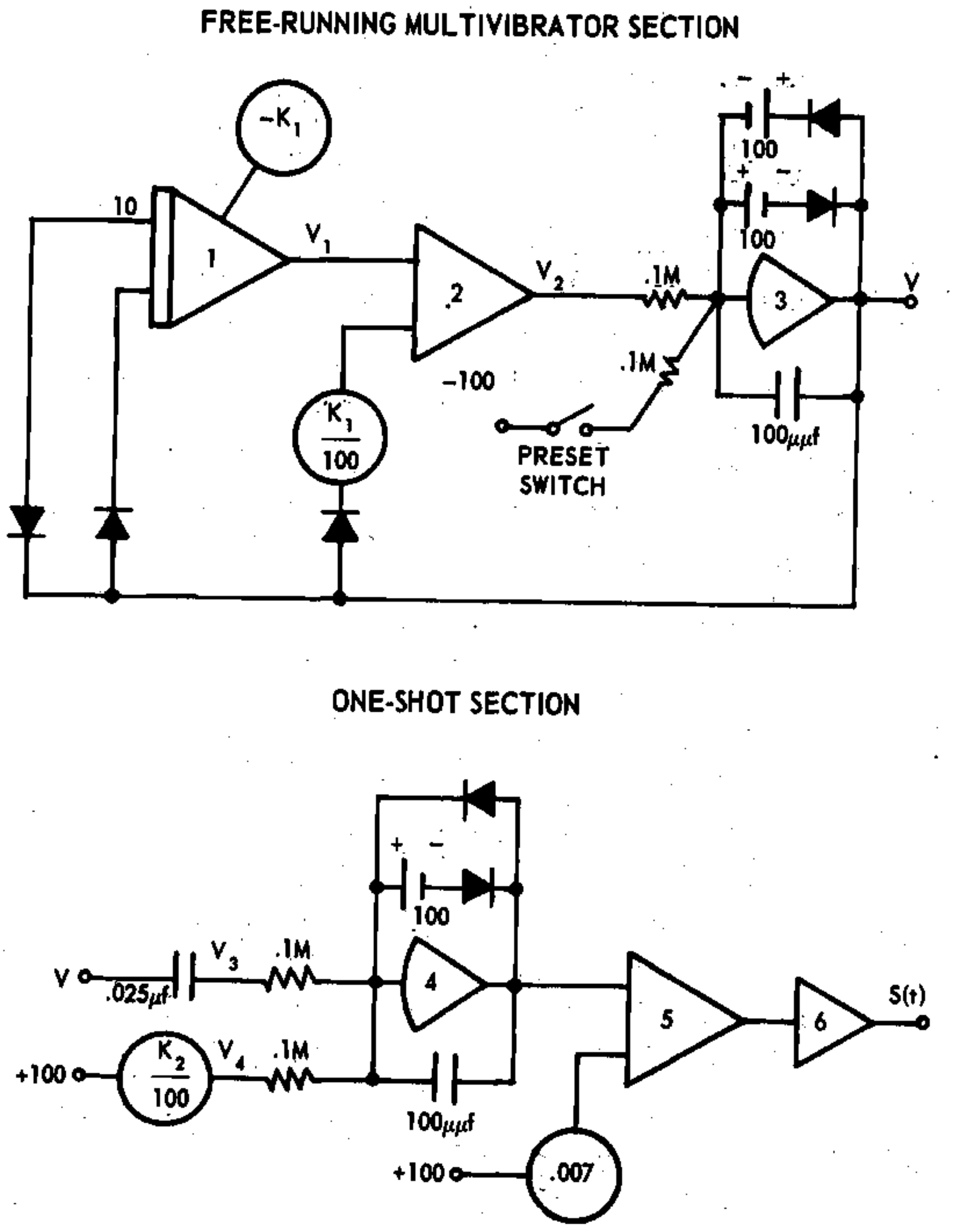

Figure 3. Analog Program for Sample Pulse Generator.

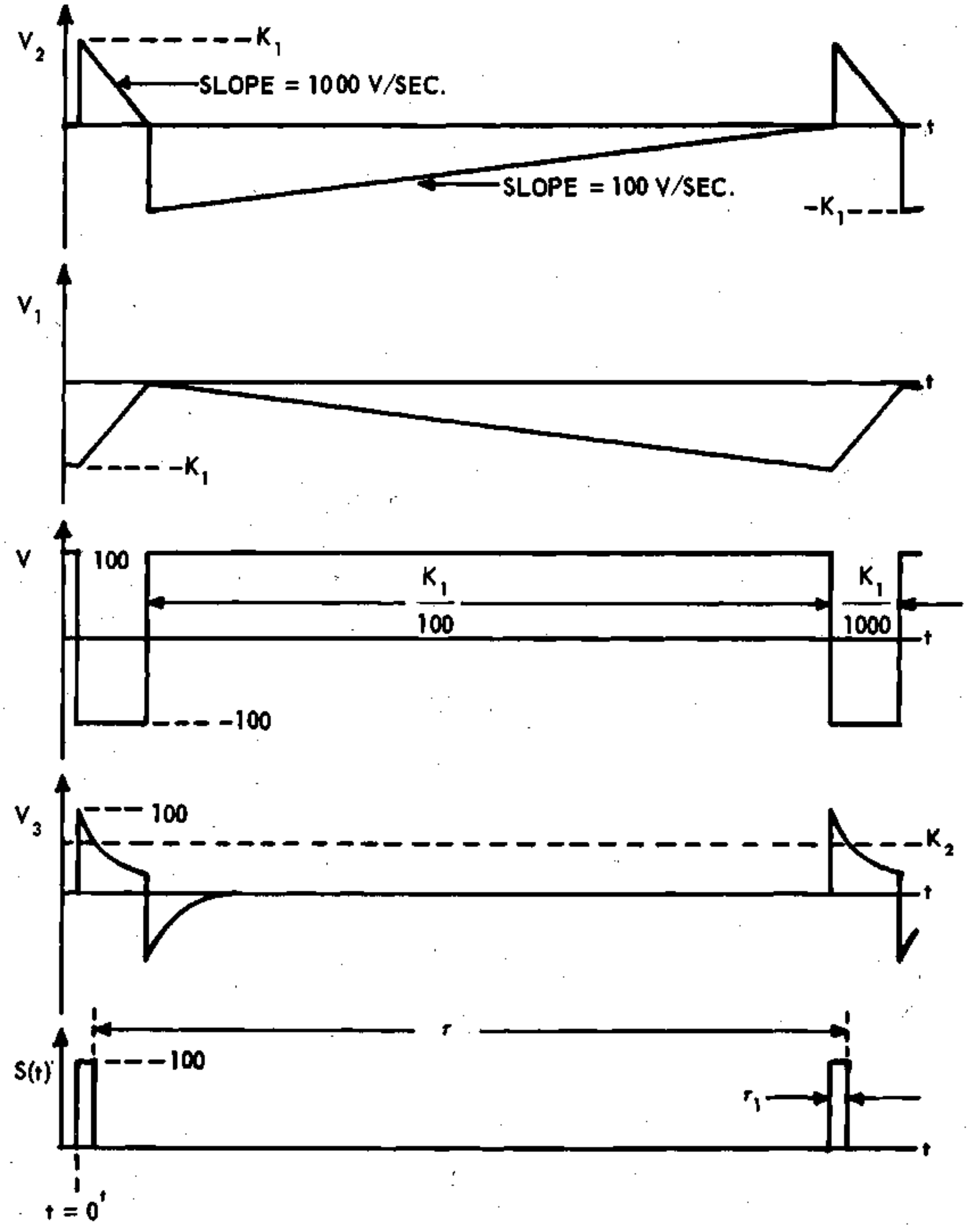

Figure 4. Sample Pulse Generator Waveforms.

occurs at $t = 0^+$. This is necessary because the sample pulses must be referenced to the time base when measuring nonstationary processes. Waveforms in Figure 4 are shown in this "preset" condition at $t = 0$.

Rise times associated with analog computer pulse circuits such as these are on the order of one millisecond. Such rise times have no effect on the operation of this system since it is never required to operate at a rate greater than four pulses per second. This upper limit on the pulse rate is based on the fact that the paper tape punch operates most reliably at rates around one sample per second or less and the fact that the serializer requires four sample pulses per sample.

This low upper limit on the sampling rate has one further ramification in that it tends to make the entire system extremely narrowband, i.e., processes applied to this system to be measured must be restricted to a bandwidth of approximately two radians per second.

System Error Analysis

The purpose of this section is to formulate the error in the measuring system calculations. This formulation may then be used in the analysis of the test results in Chapter IV. Notice that two types of errors are present in these calculations: 1) error due to inherent inaccuracy in the system itself, and 2) statistical error due to the truncation of (2.3) and (2.4) after a finite number of terms.

It is desirable to first consider the inherent system errors. The only significant errors in the system itself may be attributed to two sources: accumulative error in the sample pulse spacing and error in the digital voltmeter readings. Results of several trial runs indicate that the average error due to pulse spacing is less than 0.5 percent. The average digital voltmeter error is approximately 0.15 percent, which gives a total error of approximately 0.5 percent. As will be shown in the following analysis, the inherent system errors are much less than the statistical errors and may be neglected.

The remainder of this section is devoted to the analysis of the statistical error associated with the estimates $M_E(t_i)$ and $R_E(t_i, t_j)$ as defined by equations (2.3) and (2.4). The problem here is to formulate the error in terms of N, the number of sample functions over which the averaging described by (2.3) and (2.4) is taken, and the statistical nature of the process being measured.

$M_E(t_i)$ and $R_E(t_i, t_j)$ are random variables for any fixed pair t_i and t_j. The mean values of these random variables for fixed t_i and t_j are the true values being estimated, $M(t_i)$ and $R(t_i, t_j)$. A statistical error analysis then reduces to an analysis of the statistics of $M_E(t_i)$ and $R_E(t_i, t_j)$ about their mean values.

First consider the random variable $M_E(t_i)$. Equation (2.3) may be written as a sum of random variables

$$M_E(t_i) = \frac{\overset{1}{X}_i}{N} + \frac{\overset{2}{X}_i}{N} + \ldots\ldots + \frac{\overset{N}{X}_i}{N} \; . \tag{3.1}$$

According to Parzen (2)*, the mean and variance of (3.1) are given by

$$E[M_E(t_i)] = \frac{1}{N} E[\overset{1}{X}_i] + \frac{1}{N} E[\overset{2}{X}_i] + \ldots + \frac{1}{N} E[\overset{N}{X}_i] \tag{3.2}$$

$$\text{VAR}\ [M_E(t_i)] = \frac{1}{N^2}\ \text{VAR}\ [\overset{1}{X}_i] + \frac{1}{N^2}\ \text{VAR}\ [\overset{2}{X}_i] + \ldots + \frac{1}{N^2}\ \text{VAR}\ [\overset{N}{X}_i]. \tag{3.3}$$

*Page 407.

Since the random variables $\overset{1}{X}_i, \ldots, \overset{N}{X}_i$ have identical probability density functions, equations (3.2) and (3.3) reduce to

$$E[M_E(t_i)] = E[\overset{n}{X}_i] \tag{3.4}$$

$$\text{VAR}\,[M_E(t_i)] = \frac{1}{N}\,\text{VAR}\,[\overset{n}{X}_i]. \tag{3.5}$$

The probability density functions for the $\overset{n}{X}_i$ are not only identical to each other, but are identical to the probability density function for X_i, the vector value of $X(t)$ at $t = t_i$, since they are random samples from the infinite sequence which defines this vector. Therefore, (3.4) and (3.5) may be written as

$$E[M_E(t_i)] = E(X_i) = M(t_i) \tag{3.6}$$

$$\text{VAR}\,[M_E(t_i)] = \frac{1}{N}\,\text{VAR}\,[X_i] = \frac{1}{N}\,\sigma^2(t_i) \tag{3.7}$$

where $\sigma^2(t_i)$ is the process variance at $t = t_i$, i.e., $R(t_i, t_i)$. Equation (3.6) verifies the assertion previously made that the expected value of $M_E(t_i)$ is the true mean at $t = t_i$, $M(t_i)$. The standard deviation of $M_E(t_i)$ may be found directly from (3.7), i.e.,

$$SD[M_E(t_i)] = \sqrt{\text{VAR}[M_E(t_i)]} = \frac{\sigma(t_i)}{\sqrt{N}} \tag{3.8}$$

Equation (3.8) also represents the RMS statistical error of the estimate $M_E(t_i)$.

Now consider the random variable $R_E(t_i, t_j)$. Assuming zero mean for all time, i.e., $M(t_i) = 0$ for all t_i*, the expected value of $R_E(t_i, t_j)$ may be derived in the same manner as that for $M_E(t_i)$:

$$R_E(t_i, t_j) = \frac{\overset{1}{X}_i \overset{1}{X}_j}{N} + \frac{\overset{2}{X}_i \overset{2}{X}_j}{N} + \ . \ . \ . + \frac{\overset{N}{X}_i \overset{N}{X}_j}{N}$$

$$E[R_E(t_i, t_j)] = \frac{1}{N} E[\overset{1}{X}_i \overset{1}{X}_j] + \frac{1}{N} E[\overset{2}{X}_i \overset{2}{X}_j] + \ . \ . \ . + \frac{1}{N} E[\overset{N}{X}_i \overset{N}{X}_j]$$

$$= E[\overset{n}{X}_i \overset{n}{X}_j]$$

$$= R(t_i, t_j). \qquad (3.9)$$

This verifies the previous assertion that the expected value of the estimate $R_E(t_i, t_j)$ is the true value $R(t_i, t_j)$.

A formulation (useful for the purposes of this study) of the variance of $R_E(t_i, t_j)$ is not possible. However, some assertions can be made concerning one particular set of these estimates, in particular, that set for which $t_i = t_j$, i.e., $R_E(t_i, t_i)$. Since $R_E(t_i, t_i)$ is actually an estimate of the process variance at t_i, a more appropriate notation for this variable is $\sigma_E^2(t_i)$. Assuming that X_i is a Gaussian random variable, $\sigma_E^2(t_i)$ is chi-square distributed with N degrees of freedom and mean value $\sigma^2(t_i)$, and $\left|\sigma_E^2(t_i)\right|^{1/2} = \sigma_E(t_i)$ is chi-distributed with N degrees of freedom and mean value of $\sigma(t_i)$.

At this point it is convenient to introduce the concept of "confidence levels" as a tool for analyzing the test results found in Chapter IV.

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*This assumption is valid for all examples in this presentation.

Recall that $M_E(t_i)$ has mean $M(t_i)$, standard deviation $\sigma(t_i)/\sqrt{N}$, and has the same probability distribution as $\overset{n}{X}_i$. ($\overset{n}{X}_i$ may be considered Gaussian since this is the case for all examples used in this presentation.) Using this information, a band may be defined (in terms of $M(t_i)$, $\sigma(t_i)$, and N) into which a given percentage, say K percent, of the estimates $M_E(t_i)$ will fall. The boundaries for such a band are said to define the K percent confidence level.

For purposes of numerical illustration consider the problem of establishing an 80 percent confidence level for $M_E(t_i)$. From standard tables defining the Gaussian probability distribution, it may be found that this level is established by boundaries defined by

$$M(t_i) \pm 1.28 \frac{\sigma(t_i)}{\sqrt{N}}, \tag{3.10}$$

i.e., the band into which 80 percent of the estimates fall is 2.56 standard deviations wide and symmetrical with respect to the mean value. Note that this band may be made numerically as narrow as desired by taking N large enough. Figure 5 shows the 80 percent confidence level bounds in percent of $\sigma(t_i)$ as a function of N for M(t) estimates. Since the band is symmetrical, one curve suffices to define both the upper and lower bounds.

Similar 80 percent confidence levels may be established for the process standard deviation estimates. Recall that this random variable is chi-square distributed with N degrees of freedom and mean value $\sigma(t_i)$. Unfortunately the error band for this variable may not be defined as neatly as (3.10). However, it is possible to write an expression defining the 80 percent confidence level error bounds for the standard deviation estimates:

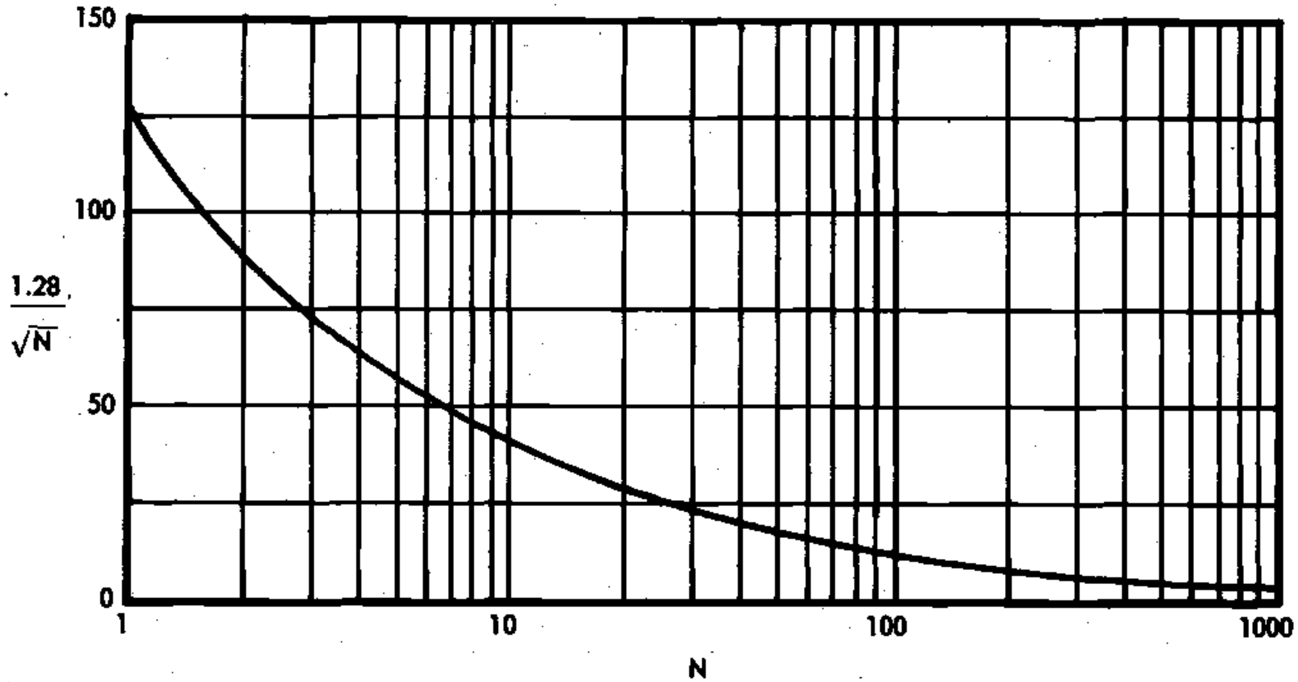

Figure 5. 80 Percent Confidence Level Error Bounds for M(t) Estimates.

$$\sigma(t_i) \begin{array}{l} + K_1\sigma(t_i) \\ - K_2\sigma(t_i) \end{array} \tag{3.11}$$

and then define K_1 and K_2 as functions of N with curves. These curves are shown in Figure 6*.

Results of this section, i.e., equations (3.10) and (3.11) and the corresponding curves in Figures 5 and 6, are used in the next chapter in the analysis of the examples presented there. For each example, once N is fixed, it is possible to go to Figures 5 and 6 and find the appropriate error bounds for the 80 percent confidence level in terms of percent of $\sigma(t_i)$.

For the remainder of the presentation, reference to the quantities $M(t_i)$ and $R(t_i, t_j)$ will imply the estimates $M_E(t_i)$ and $R_E(t_i, t_j)$ unless specifically stated otherwise.

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*Data for these curves are taken from Blackman and Tukey (3), p. 208.

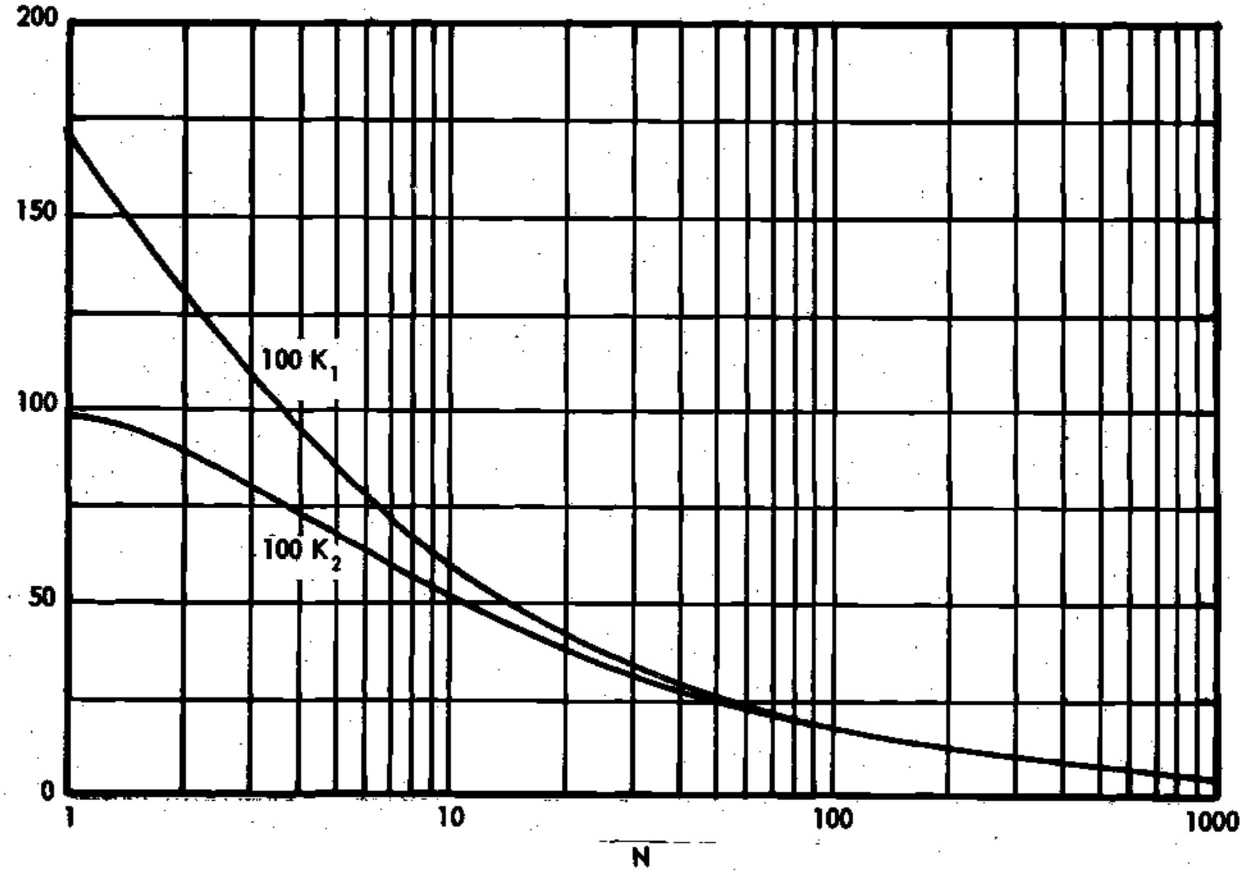

Figure 6. 80 Per Cent Confidence Level Error Bounds for $\sigma(t)$ Estimates.

CHAPTER IV

TEST RESULTS

This chapter consists of three examples which check the effectiveness of the measuring system and illustrate its use. All example processes are assumed to have a mean very close to zero, i.e., $M(t) \approx 0$ for all t^*. This is accomplished without loss of generality in testing the measuring system since the statistical error in measuring a zero-mean is the same as that for some finite mean. From equation (3.8) note that the error in the mean estimate at some t_i depends only upon $\sigma(t_i)$ and N.

Before proceeding with the examples, it is convenient at this point to describe the format in which the measuring system calculations for $R(t_i, t_j)$ are presented. The digital computer presents the covariance estimates in the form of a table, the format of which is shown in Figure 7. Since $R(t, t + \tau)$ is a function of two variables, t and $t + \tau$, the function itself is a surface and $R(t_i, t_j)$ represents discrete points on this surface. To aid in analyzing the calculated points in the following examples, it is convenient to fix one of the variables, say t_i, and observe $R(t_i, t_j)$ as a function of t_j. This corresponds to slicing the surface $R(t, t + \tau)$ at some fixed t and observing the resulting profile in the case of the continuous function. In the following examples, the calculated profiles are observed at one or more typical points so that conclusions may be drawn as to the accuracy of the calculated points.

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*It is impossible to adjust the noise generator to give a mean exactly zero.

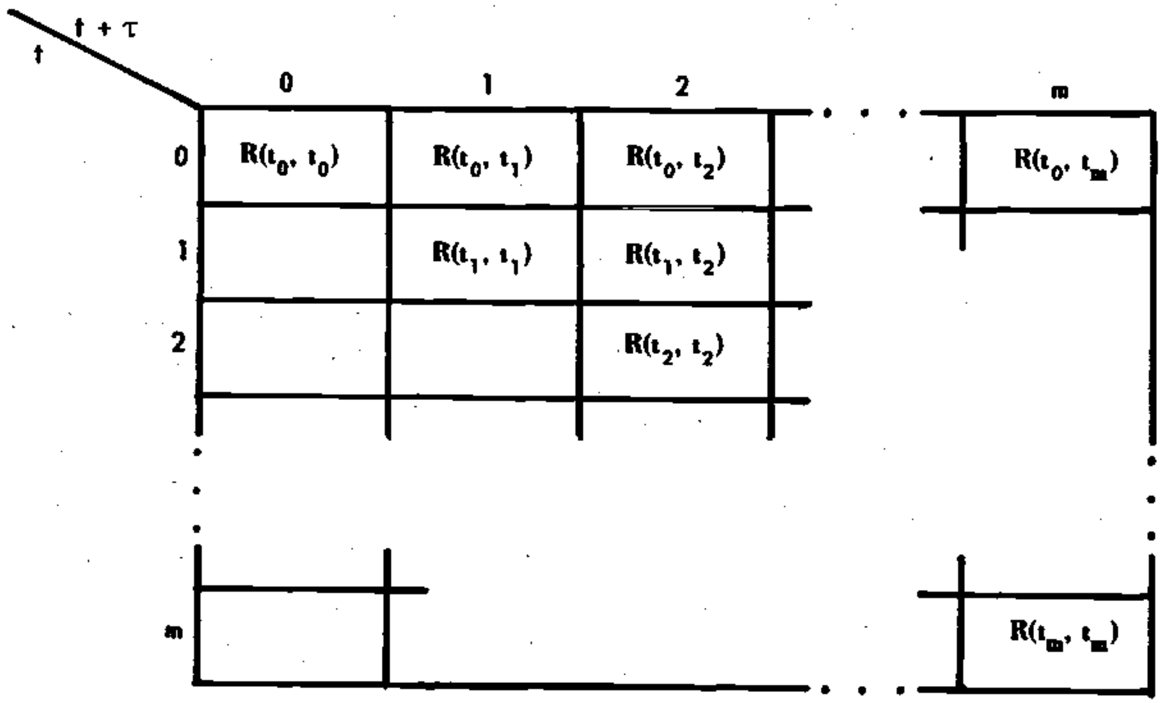

Figure 7. Format for Tabular Presentation of Covariance Estimates $R(t, t + \tau)$.

Example One. A Stationary Process

This example shows the experimental results of applying stationary filtered white noise to the measuring system previously described. The results include two cases: in the first case the averaging described by equations (2.3) and (2.4) is taken over 18 sample functions, and in the second case the averaging is taken over 180 sample functions.

First the statistical properties of the process being measured must be described analytically in order to evaluate the measuring system calculations. Consider the stochastic process X(t) obtained by passing white noise (with zero mean) through a two stage low pass filter. The transfer function for the filter is

$$H(j\omega) = \left(\frac{10}{j\omega + 2}\right)^2 .$$

Since white noise has a flat power spectrum, the power spectrum at the filter output is given by

$$G(\omega) = K \left| \left(\frac{10}{j + 2}\right)^2 \right|^2$$

where K is the power spectral density of the white noise input. For this process the covariance is a function of τ alone and can be derived using the cosine transform relating $G(\omega)$ and $R(\tau)$:

$$R(\tau) = \frac{1}{2\pi} \int_0^\infty G(\omega) \cos \omega |\tau| \, d\omega$$

$$R(\tau) = \frac{50K}{\pi} \int_{o}^{\infty} \frac{\cos \omega |\tau|}{(\omega^2 + 2^2)^2} \, d\omega$$

$$= \frac{50K}{\pi} \left[\frac{\pi}{32} (1 + 2 |\tau|) e^{-2|\tau|}\right]$$

$$= K_1 (1 + 2 |\tau|) e^{-2|\tau|} . \quad (4.1)$$

Applying this process to the measuring system yields the results shown in Figures 8 and 9 for the following measuring system parameter values: $\alpha = 1$ second, $m + 1 = 18$ samples per sample function, and $N = 18$ sample functions. In particular, Figure 8 shows the calculated process standard deviation and mean; both functions should be constant for all t_i since the process is stationary. These calculated functions appear to be highly erratic; but since the averaging in this case is taken over only 18 sample functions, the corresponding error bounds which define the 80 percent confidence level are extremely wide. (The boundaries for these error bounds are shown as dotted lines.) With respect to the error bounds, the statistical fluctuations in the calculated values are reasonable. Figure 9 shows some of the calculated points for the covariance and also the theoretical curve corresponding to $R(\tau)$ as defined by equation (4.1). The calculated points in this figure represent $R(t, t + \tau)$ as a function of τ for three fixed values of t: $t = 0, 1, 2$. For each value of τ, the three points should fall on the theoretical curve since the covariance for this case depends upon τ alone. Statistical fluctuations for this case are not unreasonable since the calculations are based on averages over only 18 sample functions.

Figures 10 and 11 show the results of applying the same process to

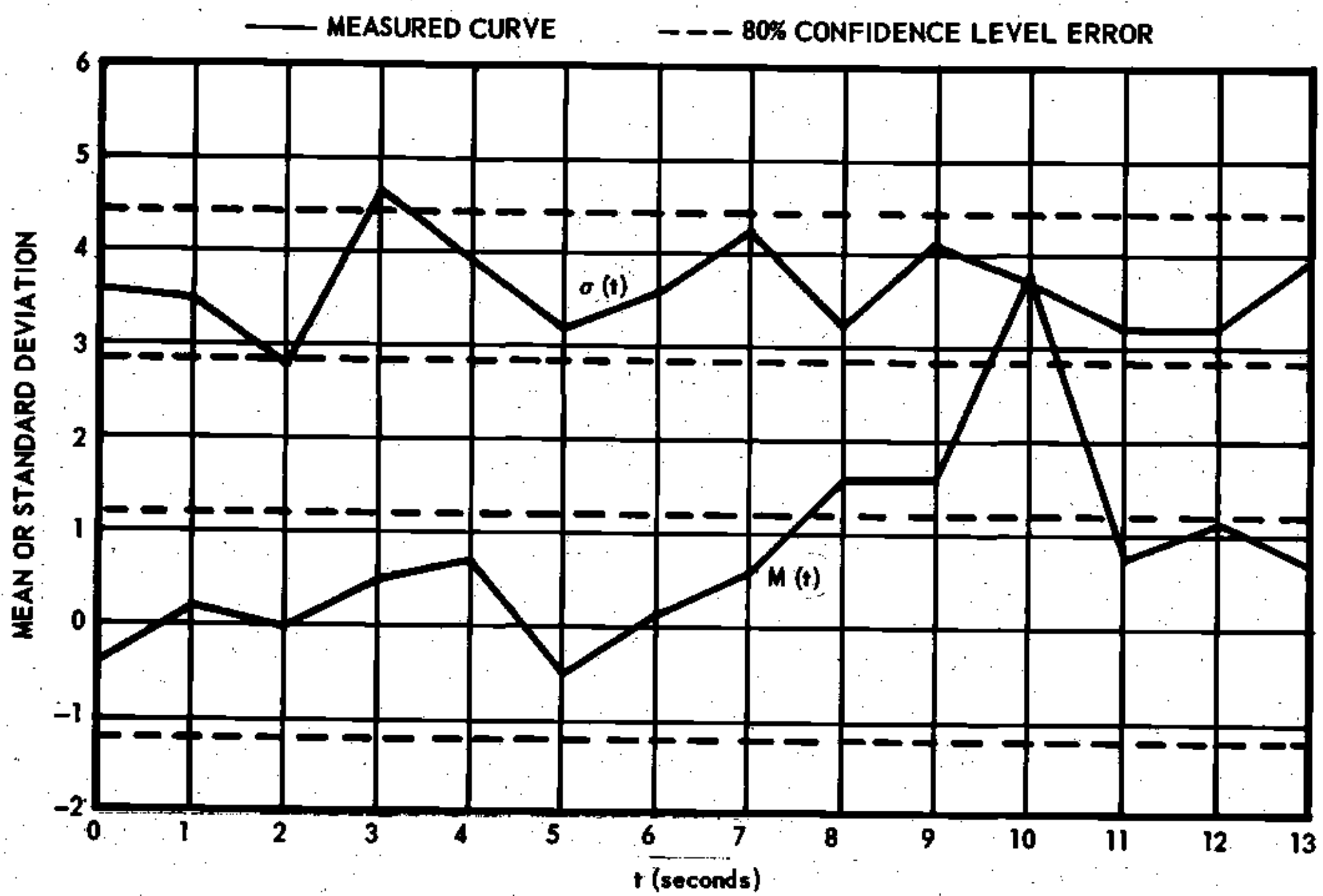

Figure 8. Mean and Standard Deviation Calculations for Example 1, N = 18.

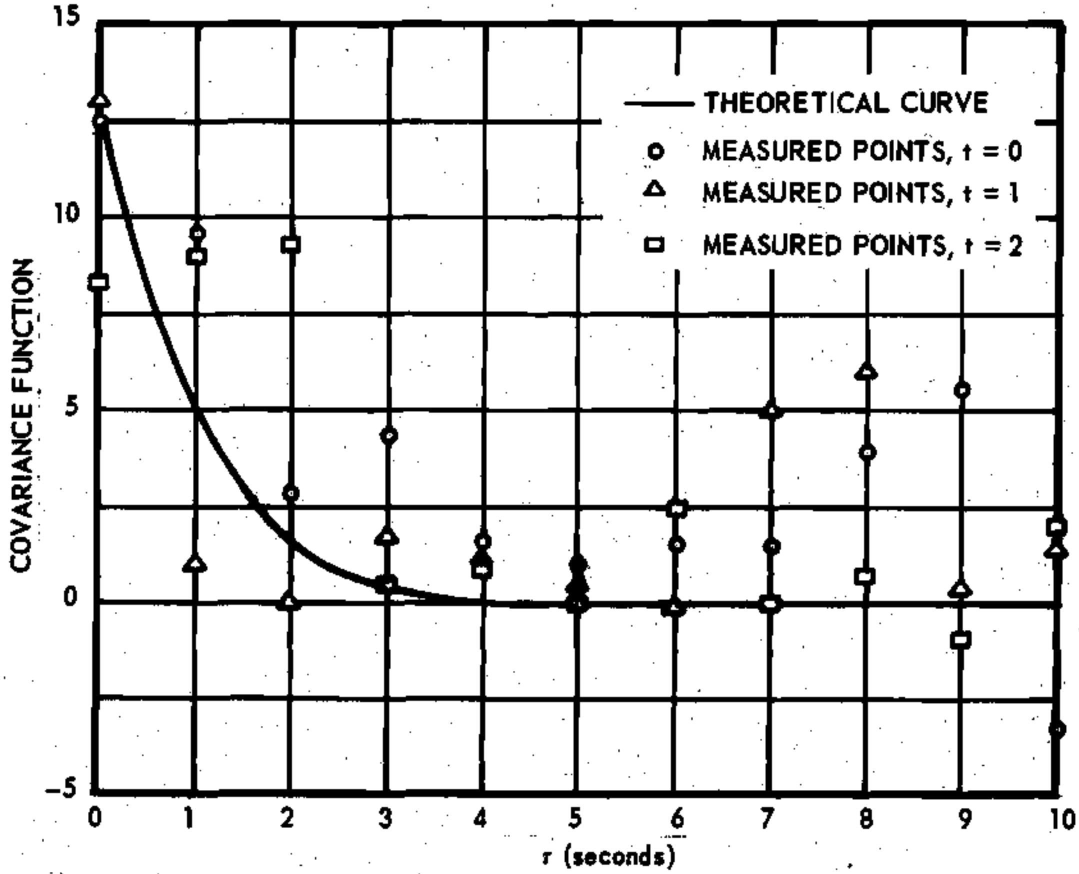

Figure 9. R(τ) Calculations with t as a Parameter for Example 1, N = 18.

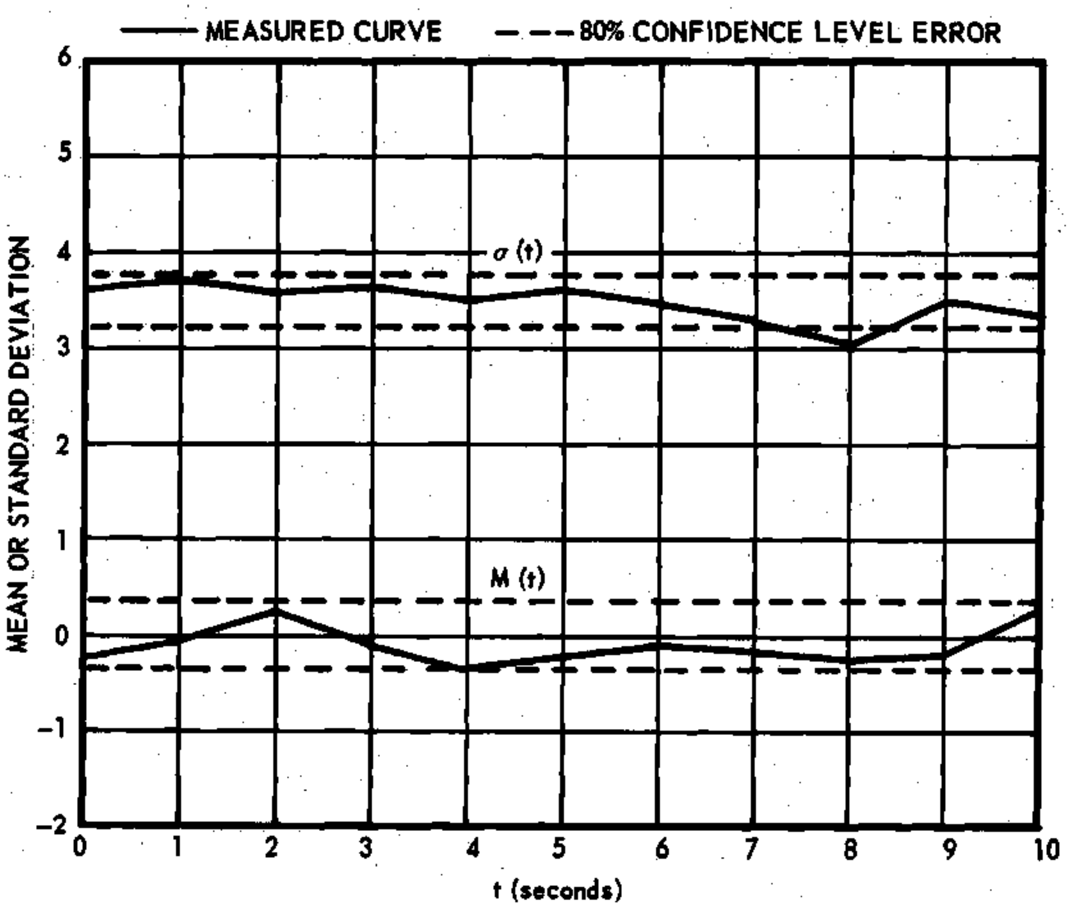

Figure 10. Mean and Standard Deviation Calculations for Example 1, N = 180.

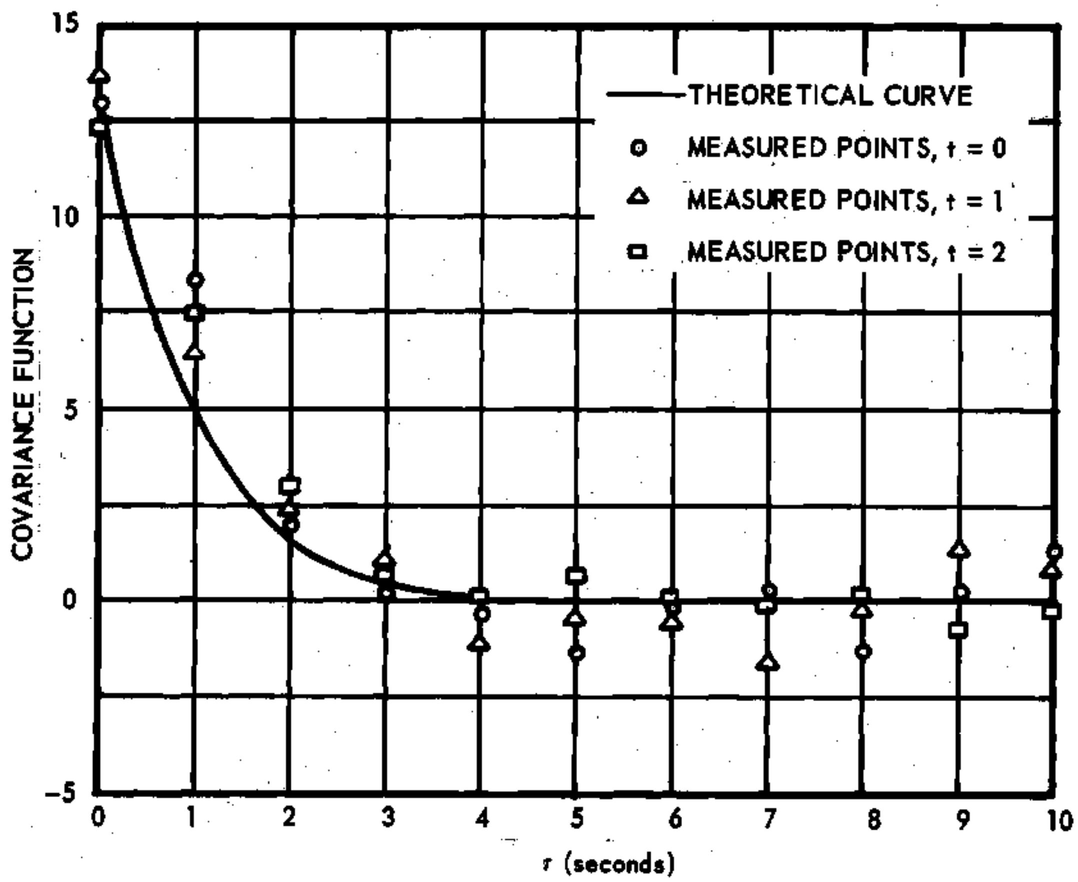

Figure 11. R(τ) Calculations with $\underset{\sim}{t}$ as a Parameter for Example 1, N = 180.

the measuring system with $N = 180$ rather than $N = 18$ as in the previous case. Information given in Figures 10 and 11 corresponds respectively to the information given in Figures 8 and 9 for the previous case. The statistical fluctuations in the calculated values for $M(t)$, $\sigma(t)$, and $R(\tau)$ as shown in these figures are much smaller, as expected. Even though the error bounds representing the 80 percent confidence level are narrower, the fluctuations in the calculated values are very reasonable with regard to these error bounds.

Example Two. A Nonstationary Process

In this example a nonstationary process is applied to the measuring system and the resulting calculations are analyzed. The nonstationary process is generated by applying white noise to a time-varying filter. (This filter is described in detail in Appendix I.) This particular process has zero mean and covariance given by

$$R(t,\ t + \tau) = \frac{K}{t + \tau} \qquad (4.2)$$

where K is the power spectral density of the white noise. The process standard deviation may be derived from the covariance as follows:

$$\sigma(t) = \sqrt{R(t,\ t + \tau)}\Big|_{\tau = o}$$

$$= \left(K/t\right)^{1/2} \qquad (4.3)$$

For this example the measuring system parameter values are $\alpha = 1$ second, $m + 1 = 10$ samples per sample function, and $N = 150$ sample functions. Figure 12 shows the measuring system calculations for $M(t)$ and $\sigma(t)$ and the corresponding 80 percent confidence level error bounds. The first row from the table of covariance calculations is shown plotted in Figure 13. This is equivalent to plotting $R(t, t + \tau)$ as a function of τ with t fixed at $t = 1$. The calculated points in Figure 12 fall within the error bounds. Since the errors associated with the calculated points for $R(t, t + \tau)$ are of the same order of magnitude as those for the calculated points for $\sigma(t)$, these points may also be assumed reasonable with respect to the given error limits.

Example 3. A More Complex Nonstationary Process

In this example a nonstationary process somewhat more complex than that in Example 2 is applied to the measuring system. Here again the process is generated by applying white noise to a time-varying filter. (This filter is also described in detail in Appendix I.) This process is statistically characterized by a zero mean and covariance given by

$$R(t, t + \tau) = \frac{Kt}{[(t + \tau) + 1][t + 1]} \quad . \tag{4.4}$$

The process standard deviation may be calculated from (4.4) as follows:

$$\sigma(t) = \sqrt{R(t, t + \tau)}\Big|_{\tau=0}$$

$$= \frac{\sqrt{Kt}}{t + 1} \quad . \tag{4.5}$$

The measuring system parameter values for this example are the same as those for Example 2. Figures 14 and 15 yield the same information for

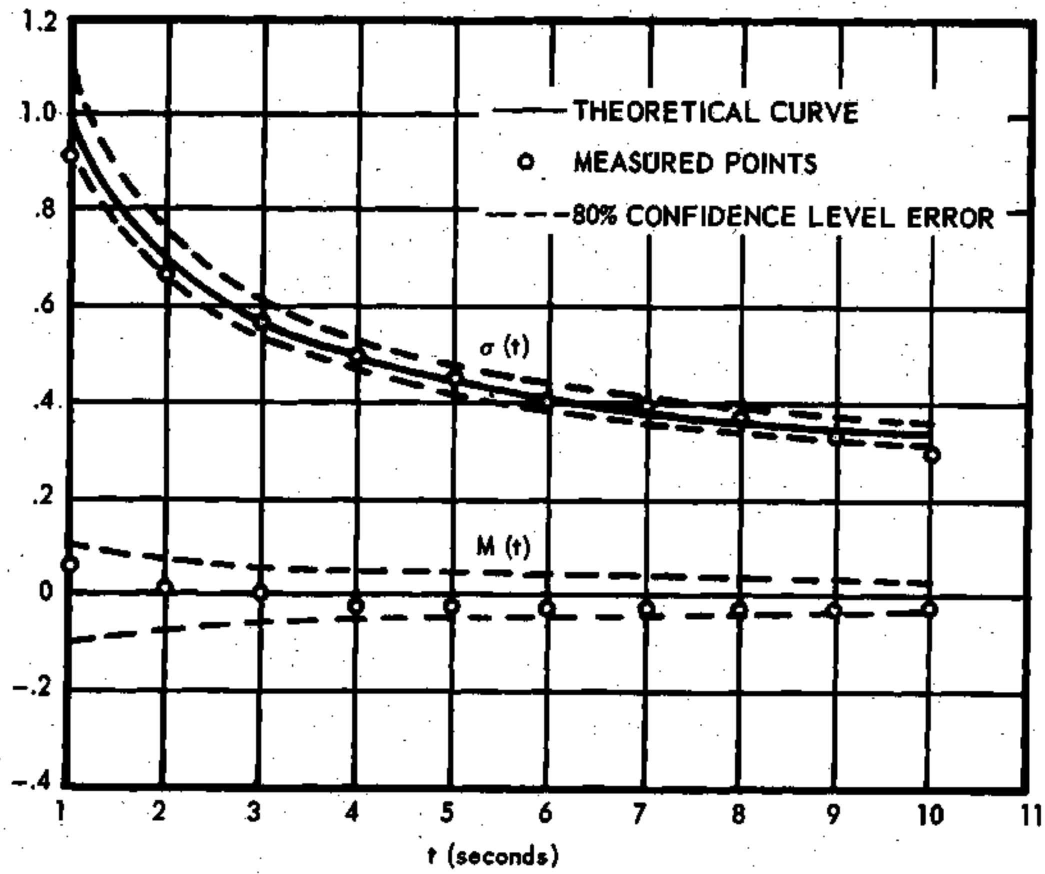

Figure 12. Mean and Standard Deviation Calculations for Example 2.

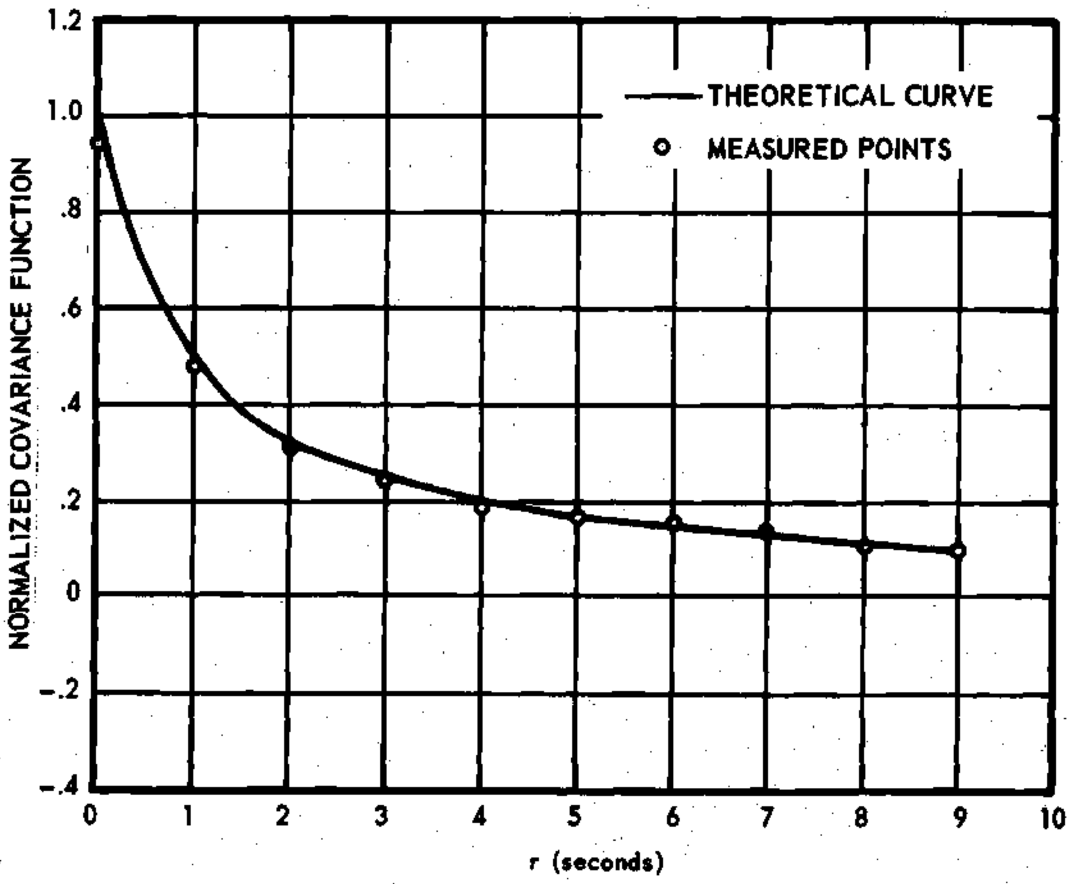

Figure 13. R(t, t + τ) Calculations with t = 1 for Example 2.

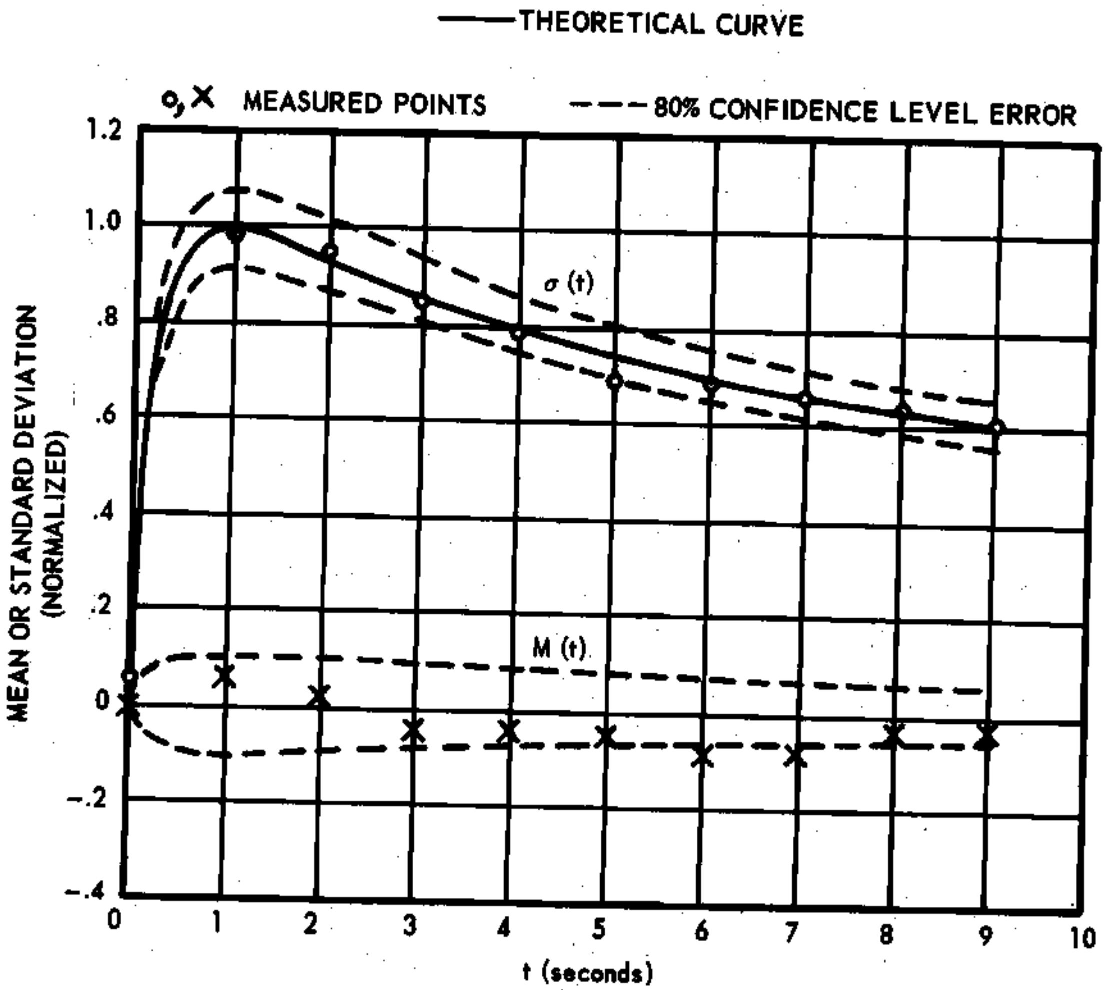

Figure 14. Mean and Standard Deviation Calculations for Example 3.

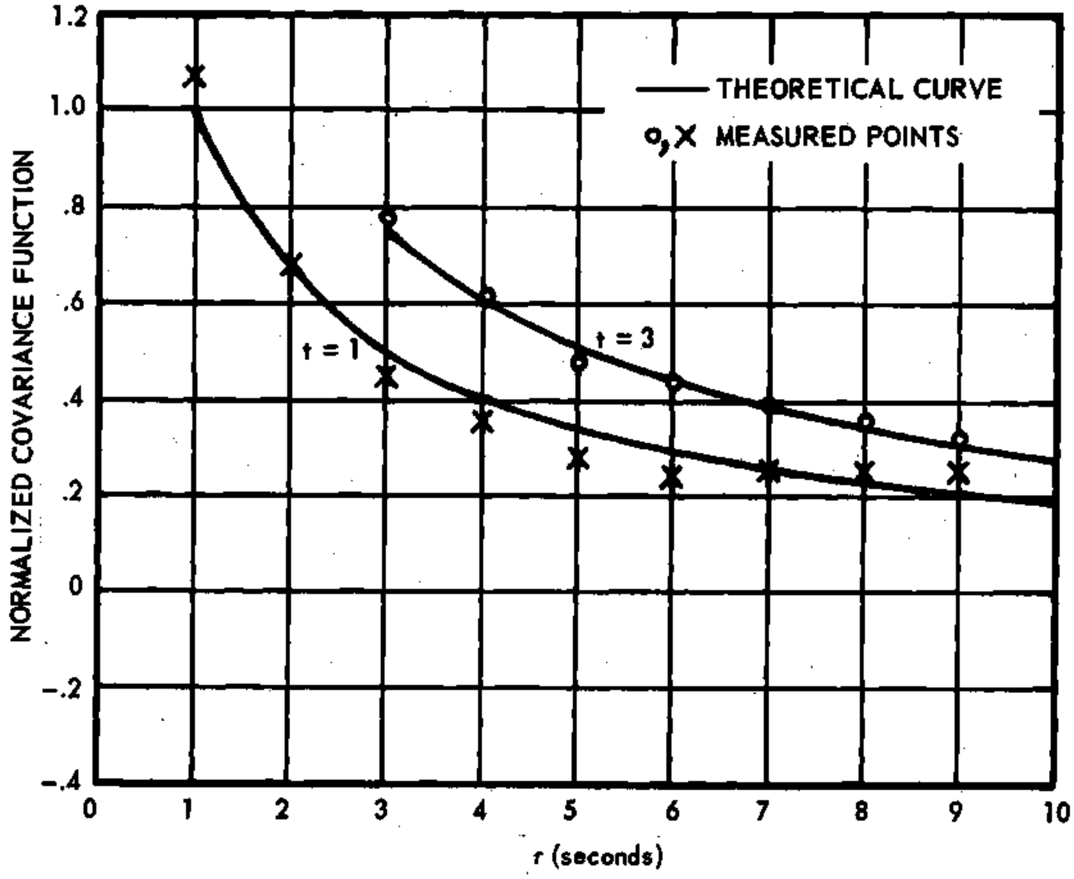

Figure 15. R(t, t + τ) Calculations with t = 1, 3 for Example 3.

this example respectively as Figures 12 and 13 for the previous example. Note one exception in that two rows of points (the first and third row) from the table of covariance calculations are shown in Figure 15. Here again the calculated points for $M(t)$, $\sigma(t)$, and $R(t, t + \tau)$ are clearly reasonable with respect to the 80 percent confidence level error bounds.

CHAPTER V

CONCLUSIONS

In the preceding chapters a system for measuring the first two moments of a nonstationary stochastic process is described. Operation of the measuring system is based on the estimates defined by equations (2.3) and (2.4) since exact calculations are impossible. Statistical error in the estimates may be described as shown in Chapter IV. Equations (3.8) and (3.10) imply that errors may be reduced to any desired level by making N large enough; however, once these errors are lowered below some given level, the inaccuracies in the measuring system itself become dominant and further increases in N are useless.

Results of applying several processes to the measuring system indicate that the system functions properly according to the requirements specified in Chapter III, and that the estimates upon which the measuring system is based lead to results which may be statistically controlled and predicted. More precisely, of the 90 points calculated with appropriate 80 percent confidence levels shown in Chapter IV, 86 percent of these points fall within the confidence level bounds.

APPENDIX I

TIME-VARYING FILTERS

The nonstationary stochastic processes used in Examples 2 and 3 are both generated by the operation of a linear, time-varying filter on stationary white noise. The following paragraphs describe the methods used to arrive at the appropriate filter for a given desired covariance function and the implementation of the filter on an analog computer.

Using the synthesis procedure of Webb (1), it is possible to translate a desired covariance function into a linear differential equation of the form

$$L\,\{X\} = N\,\{Y\} \qquad \text{(A-1.1)}$$

where L and N are linear differential operators with time-varying coefficients, Y is the white noise being operated on, and X is the desired resultant nonstationary process with prescribed covariance. Equation (A-1.1), for most cases, may be programmed on the analog computer in a straightforward manner. The analog computer model then performs the actual filtering.

Filter for Example Two

The differential equation corresponding (A-1.1) which gives the process with covariance described by (4.2) in Example 2 is given by

$$\dot{X} + \frac{1}{t}\,X = \frac{1}{t}\,Y \quad . \qquad \text{(A-1.2)}$$

This equation may be programmed on the analog computer in a straightforward manner; however, the nature of the coefficients requires that the filter be started, i.e., the computer be placed in the compute mode, with initial conditions corresponding to $t = 1$ rather than $t = 0$. Since these initial conditions are difficult to implement statistically, a different approach is used in programming a model to represent equation (A-1.2).

First it is necessary to solve (A-1.2) for X as a function of Y and t. This solution is given by

$$X = \frac{1}{t} \int_{o}^{t} Y(\tau)\ d\tau. \qquad \text{(A-1.3)}$$

Equation (A-1.3) may now be programmed in lieu of (A-1.2) and although it is still necessary to implement initial conditions corresponding to $t = 1$, the implementation is quite simple.

Consider the analog program for Equation (A-1.3) shown in Figure 16. Rather than begin each sample function with the computer "compute" switch, the "start" switch shown in Figure 16 is used. At the end of one second amplifier 1 has attained the appropriate initial condition, i.e.,

$$t\ X(1) = \int_{o}^{1} Y(\tau) d\tau,$$

and relay 1 automatically closes. This in turn grounds the computer A and B relays which places the computer in the "compute" mode and brings the rest of the program into operation. Note that the circuit used to generate 100/t begins at $t = 1$. Also note that this circuit employs two fixed log diode function generators which are standard analog computer components.

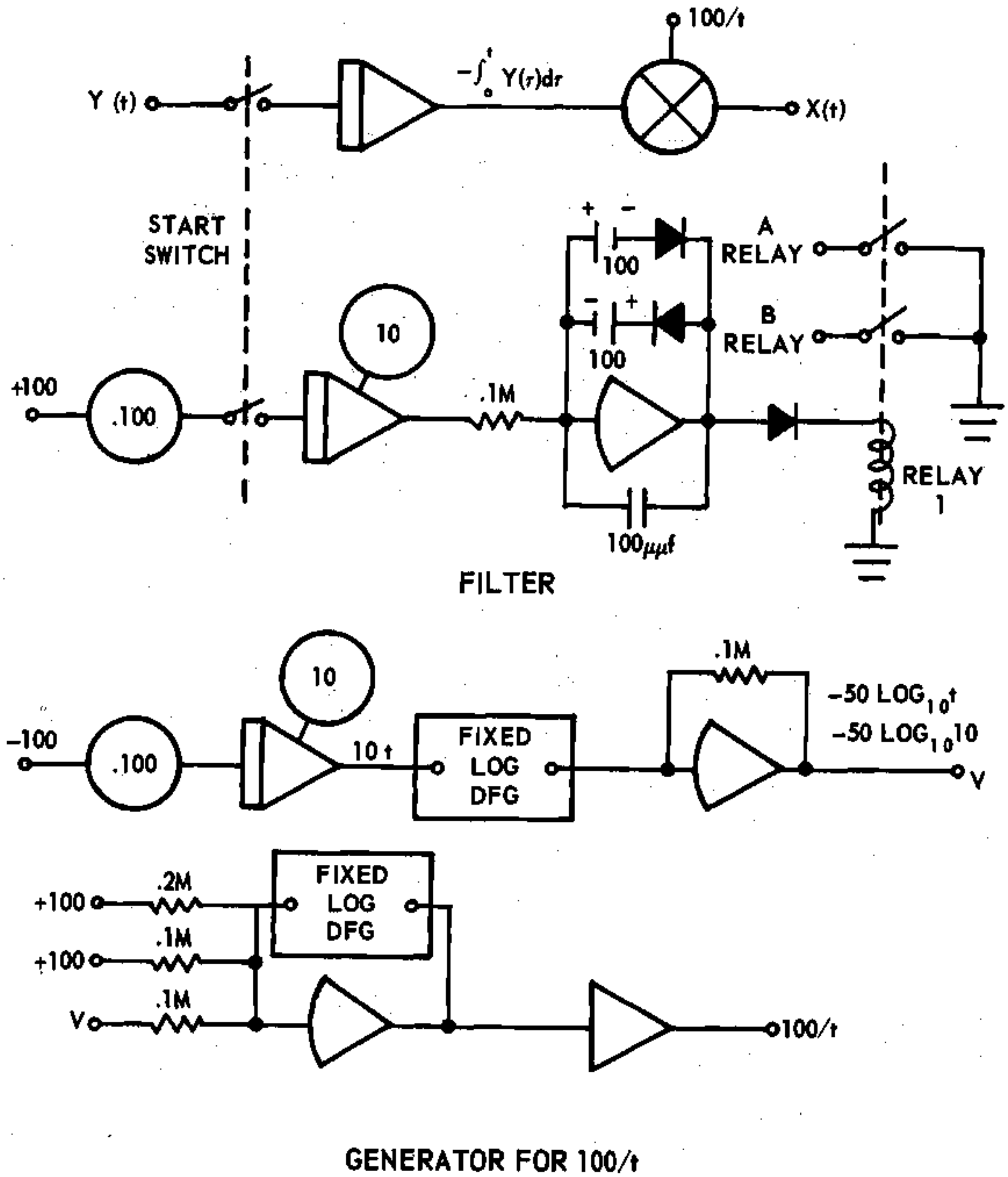

Figure 16. Analog Program for Example 2 Filter.

Filter for Example Three

Now consider the differential equation which yields the process used in Example 3 with covariance given by (4.4):

$$\mathring{X} + \frac{1}{t + 1} X = \frac{1}{t + 1} Y \ . \qquad \text{(A-1.4)}$$

This equation may be programmed directly since both the initial conditions on X, i.e., X(0) = 0, and the time-varying coefficients may be handled in a straightforward manner. The analog computer program is shown in Figure 17. Note that the circuit in Figure 16 used to generate 100/t may be used in conjunction with the program in Figure 17 to generate 100/(t + 1).

Before concluding this section, one comment on the spectral characteristics of the filters described by (A-1.3) and (A-1.4) is appropriate. These filters are in general extremely narrowband. There is a reason for this--namely, that the analog-to-digital equipment used in this experiment is extremely slow and the process being measured must be restricted to a bandwidth of approximately two radians per second. This of course in no way detracts from the generality of the procedures used to synthesize these filters.

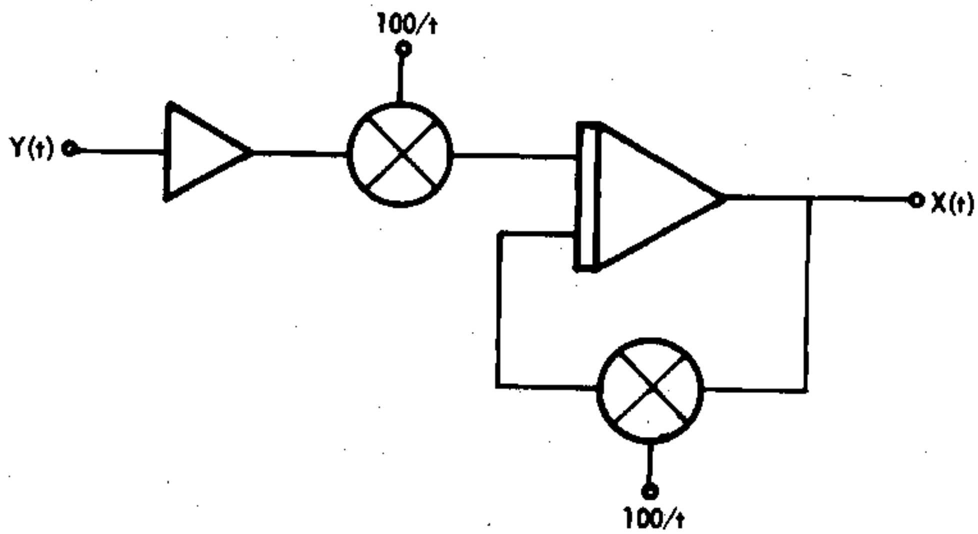

Figure 17. Analog Program for Example 3 Filter.

APPENDIX II

EQUIPMENT DESCRIPTION

In a presentation of this nature it is necessary to provide enough information about the equipment used so that another investigator could conceivably repeat the experiments described. As previously mentioned, the equipment used in these experiments is relatively standard with the exception of the analog programs which are described in Appendix I and Chapter III. Therefore, it will suffice to provide here sufficient identification for commercially available equipment and reference to descriptive reports for items not commercially available. Such a listing follows:

1) Analog Computer--Berkeley EASE Computer, Beckman Instruments, Inc., 2200 Wright Avenue, Richmond, California.

2) Gaussian Noise Generator--see "A Low-Frequency Gaussian Noise Generator for Simulation Studies," R. S. Johnson, et.al., Technical Note No. 11, Project A-366, Georgia Tech Engineering Experiment Station, Atlanta, Georgia, July 15, 1960.

3) Counter-Controller--Eagle Microflex Reset Counter model HZ40AC, Eagle Signal Company, Moline, Illinois.

4) Digital Voltmeter--Epsco model DV-803, Epsco, Inc., 275 Massachusetts Avenue, Cambridge 39, Massachusetts.

5) Serializer--see "An Analog-to-Digital Converter," R. D. Wetherington, et.al., Technical Note No. 10, Project A-366, Georgia Tech Engineering Experiment Station, Atlanta, Georgia, July 15, 1960.

6) Paper tape punch (and associated electronics)--Teletype model BRPE2 high speed perforator (with BRPEB2 base) and Teletype model BCV1 electronic control unit (with BPV1 power supply), Teletype Corp., 4100 Fullerton Avenue, Chicago, 30, Illinois.

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